ON THE SYMMETRY OF ASCENTS AND DESCENTS OVER 01-FILLINGS OF MOON POLYOMINOES

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Abstract. The purpose of this short paper is to put recent results on the symmetry of the joint distribution of the numbers of crossings and nestings of two edges over matchings, set partitions and linked partitions, in the larger context of the enumeration of increasing and decreasing chains of length 2 in fillings of moon polyominoes.

1. Introduction and Main result

1.1. **Introduction.** Two natural statistics on simple graphs with vertex set $[n]$ (or any linear order), where as usual $[n] := \{1, 2, \cdots, n\}$ for any positive integer n, are the numbers of *crossings and nestings of two edges*, also called 2-*crossings* and 2-*nestings*. Let G be a *simple graph* (no multiple edges and loops) on $[n]$. A 2-*crossing* (resp., 2-*nesting*) of G is just a sequence of two arcs $(i_1, j_1), (i_2, j_2)$ of G such that $i_1 < i_2 < j_1 < j_2$ (resp., $i_1 < i_2 < j_2 < j_1$). If we draw the vertices of G in increasing order on a line and draw the arcs above the line, crossings and nestings have the obvious geometrical meaning (see Figure 1 for an illustration). The numbers of 2-crossings and 2-nestings of G will be denoted by $\mathrm{cros}_2(\mathrm{G})$ and $\mathrm{nest}_2(\mathrm{G})$. For instance, if G is the graph represented below then $\mathrm{cros}_2(\mathrm{G}) = 4$ and $\mathrm{nest}_2(\mathrm{G}) = 6$.

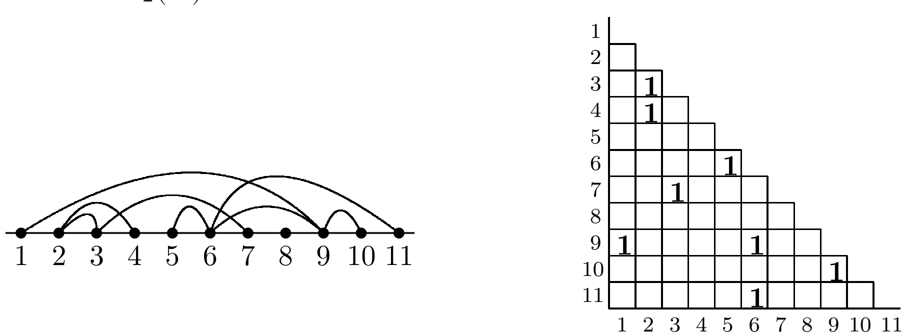


Figure 1. The graph $\{(1,9), (2,3), (2,4), (3,7), (5,6), (6,9), (6,11), (9,10)\}$ and the corresponding filling of Δ_{10}.

Recently, there has been an increasing interest in studying crossings and nestings in graphs of matchings, set partitions, linked partitions and permutations (see e.g. [1, 2, 3, 4, 5, 6, 7, 8, 10]). In particular, a property of symmetry has been systematically

established. Let $\mathcal{M}_n$, $\mathcal{P}_n$ and $\mathcal{LP}_n$ be the sets of graphs of matchings, set partitions and linked partitions of $[n]$, respectively. Then it was shown that the joint statistic $(\mathrm{cros}_2, \mathrm{nest}_2)$ is symmetrically distributed over any $\mathcal{A} \in \{\mathcal{M}_n, \mathcal{P}_n, \mathcal{LP}_n\}$ (see respectively [8, 7, 2]), i.e. for any integers k, ℓ,

$$|\{G \in \mathcal{A} : \mathrm{cros}_2(\mathrm{G}) = \mathrm{k}, \mathrm{nest}_2(\mathrm{G}) = \ell\}| = |\{\mathrm{G} \in \mathcal{A} : \mathrm{cros}_2(\mathrm{G}) = \ell, \mathrm{nest}_2(\mathrm{G}) = \mathrm{k}\}|,$$

or in other words,

$$\sum_{G \in \mathcal{A}} p^{\mathrm{cros}_2(\mathrm{G})} q^{\mathrm{nest}_2(\mathrm{G})} = \sum_{G \in \mathcal{A}} p^{\mathrm{nest}_2(\mathrm{G})} q^{\mathrm{cros}_2(\mathrm{G})}. \tag{1.1}$$

The purpose of this short paper is to put the latter results in the more general setting of fillings of arrangements of cells. Note that such a generalization (for different notions of crossing and nesting [3]) was initiated by Krattenthaler [9] and prolonged by Rubey [11] and De Mier [5, 6].

The start is the correspondence between simple graphs on $[n]$ and 01-fillings of the triangular Ferrers diagram Δ_n (in French notation) of shape $(n-1, n-2, \ldots, 1, 0)$. The correspondence consists in labeling columns from top to bottom by $\{2, 3, \ldots, n\}$ and the rows from left to right by $\{1, 2, \ldots, n-1\}$. Then put a 1 in the cell on column labeled i and row labeled j if and only if (i, j) is an arc of G. An illustration is given in Figure 1. In this correspondence, crossings are send on *descents* and nestings on *ascents*. An *ascent*, or *North-East chain of length* 2, (resp., *descent* or *South-East chain of length* 2) in a 01-filling is just a set of two 1's in the filling such that one of them is above and to the right (resp., below and to the left) of the other and the smallest rectangle contained the two 1's is contained in the diagram. Then (1.1) can be translated into a property of symmetry of ascents and descents over 01-fillings of the triangular shape Δ_n according to some restrictions on the numbers of 1's in columns and rows. In this paper, we establish such a property in more general arrangements, namely in moon polyominoes.

1.2. **The main result.** A *polyomino* is an arrangement of square cells. It is *convex* if along any row of cells and along any column of cells there is no hole. It is *intersection free* if any two rows are comparable, i.e., one row can be embedded in the other by applying a vertical shift. A *moon polyomino* is a convex and intersection free polyomino. An illustration is given below.

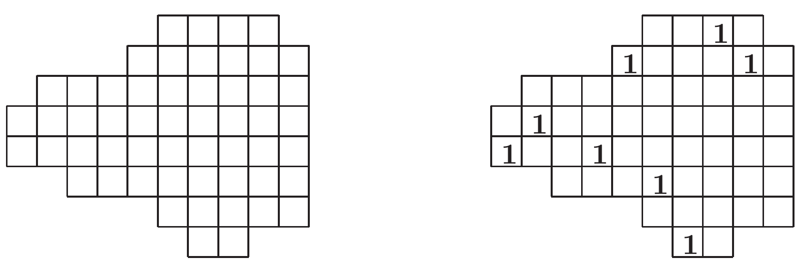


FIGURE 2. A moon polyomino T and a filling in $\mathcal{N}(T, \mathbf{m}; A)$, with $\mathbf{m} =$ $(1, 2, 0, 1, 2, 1, 0, 1)$ and $A = \{3, 10\}$.

Let T be a moon polyomino with s rows and t columns. By convention, we always label the rows from top to bottom by $\{1, 2, \ldots, s\}$ and the columns of T from left to right by $\{1, 2, \ldots, t\}$. A 01-*filling* F of T consists of assigning 0 or 1 to each cell. We say that a cell is *empty* if it has been assigned the value 0. Given a s-uple $\mathbf{m} = (m_1, \ldots, m_s)$ (resp., $\mathbf{m} = (m_1, \ldots, m_t)$) of integers, we denote by $\mathcal{N}(T, \mathbf{m})$ (resp., $\mathcal{N}'(T, \mathbf{m})$) the set of 01-fillings of T with exactly m_i 1's in row (resp., column) labelled i such that there is at most one 1 in each column (resp., row). Also, if A is a set of positive integers, we denote by $\mathcal{N}(T, \mathbf{m}; A)$ (resp., $\mathcal{N}'(T, \mathbf{m}; A)$) the set of fillings F in $\mathcal{N}(T, \mathbf{m})$ (resp., $\mathcal{N}'(T, \mathbf{m})$ whose set of the indices of its empty columns (resp. rows), denoted $\mathrm{EC(F)}$ (resp., $\mathrm{ER(F)}$), is equal to A. An example is given in Figure 2. Also, if F is a 01-filling, we denote by $\mathrm{ne_2(F)}$ and $\mathrm{se_2(F)}$ the number of ascents and descents in F. For instance, if F is the filling of Figure 1, we have $\mathrm{ne_2(F)} = 6$ and $\mathrm{se_2(F)} = 4$, while for the filling of Figure 2 we have $\mathrm{ne_2(F)} = \mathrm{se_2(F)} = 4$. We can now state the main result of the paper, which is a generalization of (1.1).

Theorem 1.1. *For any moon polyomino T, the joint statistic* $(\mathrm{ne_2}, \mathrm{se_2})$ *is symmetrically distributed over each $\mathcal{N}(T, \mathbf{m}; A)$ and each $\mathcal{N}'(T, \mathbf{m}; A)$, i.e., we have*

$$\sum_{F \in \mathcal{N}(T, \mathbf{m}; A)} p^{\mathrm{ne_2(F)}} q^{\mathrm{se_2(F)}} = \sum_{F \in \mathcal{N}(T, \mathbf{m}; A)} p^{\mathrm{se_2(F)}} q^{\mathrm{ne_2(F)}},$$

$$\sum_{F \in \mathcal{N}'(T, \mathbf{m}; A)} p^{\mathrm{ne_2(F)}} q^{\mathrm{se_2(F)}} = \sum_{F \in \mathcal{N}'(T, \mathbf{m}; A)} p^{\mathrm{se_2(F)}} q^{\mathrm{ne_2(F)}}.$$

Summing over all A in (3.2), we get the following.

Theorem 1.2. *For any moon polyomino T, the joint statistic* $(\mathrm{ne_2}, \mathrm{se_2})$ *is symmetrically distributed over each $\mathcal{N}(T, \mathbf{m})$ and $\mathcal{N}'(T, \mathbf{m})$.*

The paper is organized as follows. In Section 2, we give a bijective proof of Theorem 1.1. In section 3, we give an alternative proof of the latter result by computing the distribution of the joint statistic $(\mathrm{ne_2}, \mathrm{se_2})$ over each $\mathcal{N}(T, \mathbf{m}, A)$. In section 4, we show how we can recover (1.1) from our result. Finally, we conclude this paper with some remarks.

2. Proof of Theorem 1.1

Since the transpose of a moon polyomino is always a moon polyomino, it suffices to prove the first equation of Theorem 1.1. Our first proof is bijective. Let T be a moon polyomino with s rows and t columns, $\mathbf{m} = (m_1, m_2, \ldots, m_s)$ a s-uple of nonnegative integers. We will construct an involution $\Phi : \mathcal{N}(T, \mathbf{m}) \to \mathcal{N}(T, \mathbf{m})$ such that for any $F \in \mathcal{N}(T, \mathbf{m})$, we have

$$\mathrm{EC}(\Phi(\mathrm{F})) = \mathrm{EC(F)},\ \mathrm{ne_2}(\Phi(\mathrm{F})) = \mathrm{se_2(F)},\ \mathrm{se_2}(\Phi(\mathrm{F})) = \mathrm{ne_2(F)}.$$

2.1. **Notations.** The *length-row sequence* of T, denoted $r(T)$, is the sequence $(r_1, r_2, \ldots, r_s)$ where r_i is the length of R_i, the i-th row from top. Clearly, the length-row sequence of any moon polyomino is always unimodal. Thus there exists an unique i_0 such that $r_1 \le r_2 \le \cdots \le r_{i_0} > r_{i_0} + 1 \ge \cdots \ge r_s$. The *upper part of* T, denoted $Up(T)$, is the set of rows R_i with $1 \le i \le i_0$, and the *lower part*, denoted $Low(T)$, the set of rows R_i, $i_0 + 1 \le i \le s$. For instance, if T is the moon polyomino in Figure 3, we have

$r(T) = (4, 6, 9, 10, 10, 8, 5, 2)$, $Up(T) = \{R_1, R_2, R_3, R_4, R_5\}$ and $Low(T) = \{R_6, R_7, R_8\}$. We order the rows of T by the (total) order $\prec$ defined by $R_i \prec R_j$ if and only if

- $r_i < r_j$ or
- $r_i = r_j$, $R_i \in Up(T)$ and $R_j \in Low(T)$, or
- $r_i = r_j$, $R_i, R_j \in Up(T)$ and R_i is above R_j, or
- $r_i = r_j$, $R_i, R_j \in Low(T)$ and R_i is below R_j.

It is clear that $\prec$ is a total order on the rows of T. For instance, if T is the moon polyomino in Figure 3 we have $R_8 \prec R_1 \prec R_7 \prec R_2 \prec R_6 \prec R_3 \prec R_4 \prec R_5$.

For i an integer, $1 \leq i \leq s$, the *i-th rectangle of* T, is the greatest rectangle contained in T whose top (resp., bottom) row is R_i if $R_i \in Up(T)$ (resp., $R_i \in Low(T)$). An illustration is given in Figure 3.

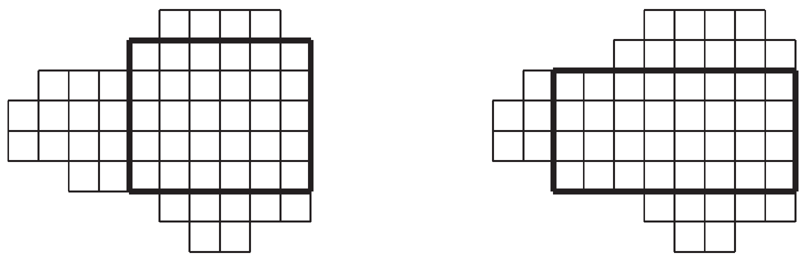


FIGURE 3. *left*: the 2-th rectangle, *right*: the 6-th rectangle.

2.2. **Coloring of fillings.** Let $F \in \mathcal{N}(T, \mathbf{m})$. The coloring of F is the colored filling obtained from F by:

- coloring the cells of the empty columns,
- for each $R_i \in Up(T)$ (resp., $R_i \in Low(T)$), coloring the cells which are contained in the i-th rectangle and below (resp., above) each $\mathbf{1}$ in R_i. An illustration is given in Figure 4.

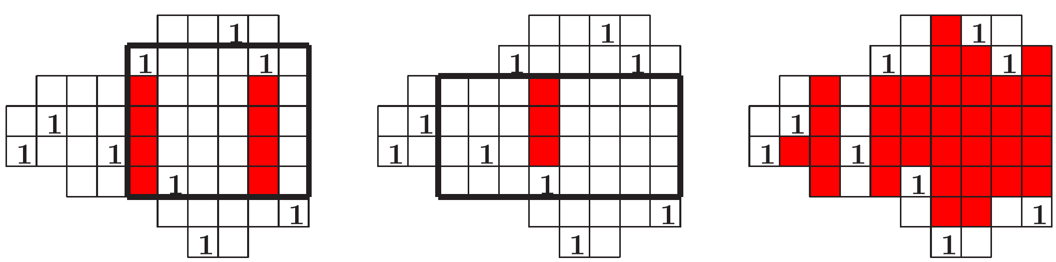


FIGURE 4. *left*: coloring induced by R_2, *center*: coloring induced by R_6, *right*: full coloring.

The interest of coloring a 01-filling is in the following result. Let c be a cell of F. If c is filled with 1 we denote by $\mathrm{luc}(c; F)$ (resp., $\mathrm{ruc}(c; F)$) the numbers of uncolored cells to the

left (resp. right) and in the same row than the cell c in the coloring of F. If c is empty, we set $\mathrm{luc(c;F)} = \mathrm{ruc(c;F)} = 0$. The proof of the following result is left to the reader.

Proposition 2.1. *Let $F \in \mathcal{N}(T, \mathbf{m})$ and c be a cell of R_i filled with $\mathbf{1}$. Then* $\mathrm{luc(c;F)}$ *(resp.,* $\mathrm{ruc(c;F)}$*) is equal to*

- *if $R_i \in Up(T)$: the number of ascents (resp., descents) contained in the i-th rectangle of F whose North-east (resp., North-west) 1 is in c,*
- *if $R_i \in Low(T)$: the number of descents (resp. ascents) contained in the i-th rectangle of F whose South-east (resp., South-west) 1 is in c.*

It then follows from the above Proposition that

$$\mathrm{ne}_2(\mathrm{F}) = \sum_{\mathrm{c} \in \mathrm{Up(F)}} \mathrm{luc(c;F)} + \sum_{\mathrm{c} \in \mathrm{Low(F)}} \mathrm{ruc(c;F)}, \tag{2.1}$$

$$\mathrm{se}_2(\mathrm{F}) = \sum_{\mathrm{c} \in \mathrm{Up(F)}} \mathrm{ruc(c;F)} + \sum_{\mathrm{c} \in \mathrm{Low(F)}} \mathrm{luc(c;F)}. \tag{2.2}$$

We can now describe our involution.

2.3. **The involution Φ.** Let $F \in \mathcal{N}(T, \mathbf{m})$. We construct $\Phi(F) \in \mathcal{N}(T, \mathbf{m})$ by the following process. We start with the polyomino (empty filling) T.

(1) Color the columns of T indexed by $\mathrm{EC(F)}$. We denote by F'_0 the result.

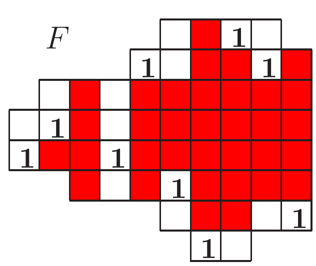
(2) Suppose $R_{i_1} \prec R_{i_2} \prec \cdots \prec R_{i_s}$. For j from 1 to s, the (colored) filling F'_j is obtained from F'_{j-1} by the following process:

- if $m_{i_j} = 0$, then do nothing,
- else, read the m_{i_j} 1's in the i_j-th row of F from left to right and denote the number of uncolored cells strictly
  - to the left of the first 1 by h_0,
  - between the u-th 1 and the $(u+1)$-th 1, $1 \le u \le m_{i_j} - 1$, by h_u,
  - to the right of the last 1 by $h_{m_{i_j}}$.

  Then fill the i_j-th row of F'_{j-1} with m_{i_j} 1's in such a way that the number of uncolored cells strictly
  - to the left of the first 1 is $h_{m_{i_j}}$,
  - between the u-th 1 and the $(u+1)$-th 1, $1 \le u \le m_{i_j} - 1$, is $h_{m_{i_j} - u}$,
  - to the right of the last 1 is h_0.

  Next, color the cells which are below (resp., above) the new 1's and contained in the i_j-th rectangle if $R_{i_j} \in Up(T)$ (resp., $R_{i_j} \in Low(T)$).

(3) Set $\Phi(F) = F'_s$. For a better understanding, we give an illustration. Suppose F is the filling given below.



Then the step-by-step construction of $\Phi(F)$ goes as follows.

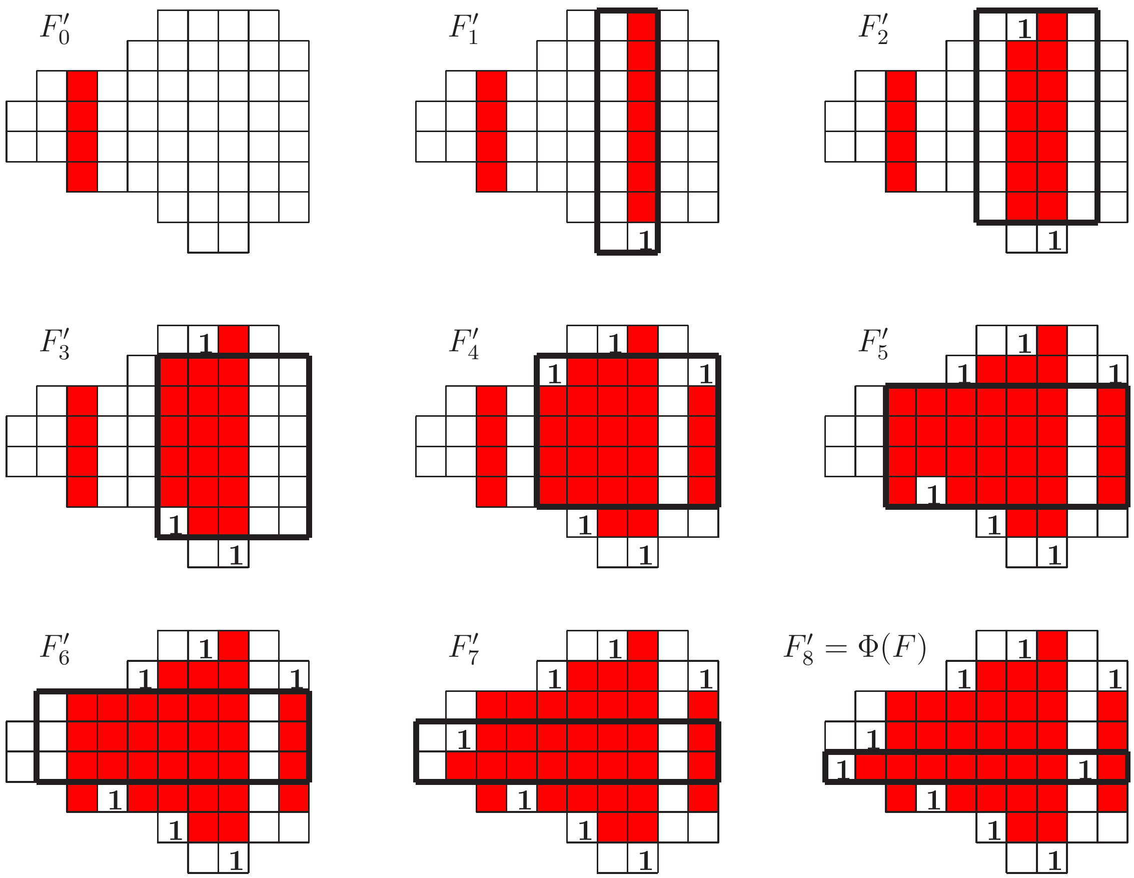


Figure 5. The step-by-step construction of $\Phi(F)$

It is easy to check that the map Φ is an involution on $\mathcal{N}(T, \mathbf{m})$ and $\mathrm{EC}(\Phi(\mathrm{F})) = \mathrm{EC}(\mathrm{F})$ for any F. Invoking Equations (2.1) and (2.2) lead to the following result which imply Theorem 1.1.

Proposition 2.2. *For any moon polyomino T, the map Φ is an involution on $\mathcal{N}(T, \mathbf{m})$ such that for any $F \in \mathcal{N}(T, \mathbf{m})$ we have*

$$\mathrm{EC}(\Phi(\mathrm{F})) = \mathrm{EC}(\mathrm{F})\ ,\ \mathrm{ne}_2(\Phi(\mathrm{F})) = \mathrm{se}_2(\mathrm{F})\ ,\ \mathrm{se}_2(\Phi(\mathrm{F})) = \mathrm{ne}_2(\mathrm{F}).$$

One can also give an alternating proof of Theorem 1.1 by computing the joint distribution of $(\mathrm{ne}_2, \mathrm{se}_2)$. This is the content of the following section.

3. Distribution of ascents and descents over $\mathcal{N}(T, \mathbf{m}, A)$

For nonnegative integers n and k, let $\left[{n \atop k} \right]_{p,q}$ be the p, q-Gaussian coefficient defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}!\,[n-k]_{p,q}!},$$

where, as usual in p,q-theory, the p,q-integer $[r]_{p,q}$ is given by

$$[r]_{p,q} := \frac{p^i - q^i}{p - q} = (p^{i-1} + p^{i-2}q + \cdots + p^j q^{i-j-1} + \cdots + pq^{i-2} + q^{i-1}),$$

and the p,q-factorial $[r]_{p,q}!$ by $[r]_{p,q}! := \prod_{i=1}^{r} [i]_{p,q}$.

Let T be a moon polyomino with s rows and t columns, $\mathbf{m} = (m_1, \ldots, m_s)$ a s-uple of nonnegative integers and A a subset of $[t]$. Suppose $R_{i_1} \prec R_{i_2} \prec \cdots \prec R_{i_s}$. Then for $j \in [s]$, define h_{i_j} by

$$h_{i_j} = r_{i_j} - (m_{i_1} + m_{i_2} + \cdots + m_{i_{j-1}}) - a_{i_j}, \tag{3.1}$$

where r_{i_j} is the length of row R_{i_j} and a_{i_j} is the number of indices $k \in A$ such that the column labeled k intersect the row labeled i_j. Then we have the following result.

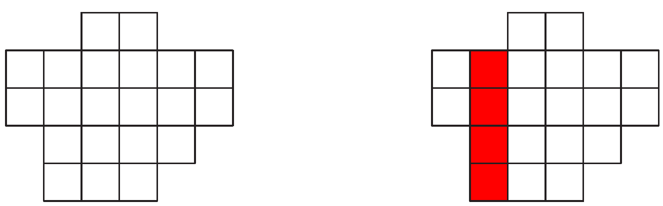
Theorem 3.1. *For any moon polyomino T, the distribution of the joint statistic* $(\mathrm{ne}_2, \mathrm{se}_2)$ *over $\mathcal{N}(T, \mathbf{m}; A)$ is given by*

$$\sum_{F \in \mathcal{N}(T,\mathbf{m};A)} p^{\mathrm{ne}_2(\mathrm{F})} q^{\mathrm{se}_2(\mathrm{F})} = \prod_{j=1}^{s} \begin{bmatrix} h_j \\ m_j \end{bmatrix}_{p,q}, \tag{3.2}$$

where h_j is defined by (3.1)*. In particular,*

$$|\mathcal{N}(T, \mathbf{m}; A)| = \prod_{j=1}^{s} \binom{h_j}{m_j}, \tag{3.3}$$

For instance, suppose T be the moon polyominoe given below and $A = \{2\}$.



Then we have:

- $R_1 \prec R_5 \prec R_4 \prec R_2 \prec R_3$ thus $i_1 = 1$, $i_2 = 5$, $i_3 = 4$, $i_4 = 2$, $i_5 = 3$.
- The column labeled 2 intersect the rows labeled $2, 3, 4, 5$, thus $a_1 = 0$, $a_2 = a_3 = a_4 = a_5 = 1$.

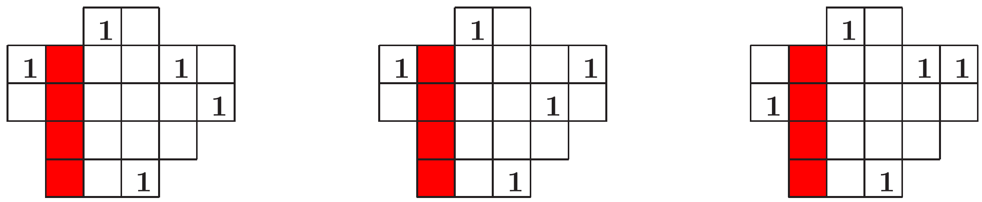
Suppose $\mathbf{m} = (1, 2, 1, 0, 1)$. We then have

$$\begin{aligned}
h_{i_1} &= h_1 = r_1 - a_1 = 2, \\
h_{i_2} &= h_5 = r_5 - m_1 - a_5 = 1, \\
h_{i_3} &= h_4 = r_4 - (m_1 + m_5) - a_4 = 1, \\
h_{i_4} &= h_2 = r_2 - (m_1 + m_5 + m_4) - a_2 = 3, \\
h_{i_5} &= h_3 = r_3 - (m_1 + m_5 + m_4 + m_2) - a_3 = 1.
\end{aligned}$$

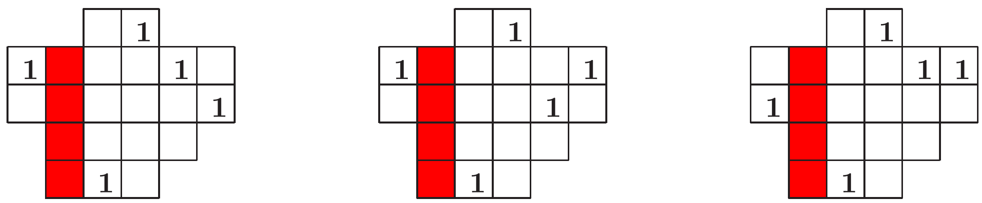
It follows that

$$\prod_{j=1}^{5} \begin{bmatrix} h_j \\ m_j \end{bmatrix}_{p,q} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{p,q} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{p,q} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{p,q} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{p,q} = p^3 + 2p^2q + 2pq^2 + q^3.$$

On the other hand, the fillings in $\mathcal{N}(T, \mathbf{m}, A)$ and the corresponding values of ne_2 and se_2 are listed below.



$\mathrm{ne}_2 = 0$, $\mathrm{se}_2 = 3$ $\quad$ $\mathrm{ne}_2 = 1$, $\mathrm{se}_2 = 2$ $\quad$ $\mathrm{ne}_2 = 2$, $\mathrm{se}_2 = 1$



$\mathrm{ne}_2 = 1$, $\mathrm{se}_2 = 2$ $\quad$ $\mathrm{ne}_2 = 2$, $\mathrm{se}_2 = 1$ $\quad$ $\mathrm{ne}_2 = 3$, $\mathrm{se}_2 = 0$

Summing up we get $\sum_{F \in \mathcal{N}(T,\mathbf{m},A)} p^{\mathrm{ne}_2(\mathrm{F})} q^{\mathrm{se}_2(\mathrm{F})} = p^3 + 2p^2q + 2pq^2 + q^3$, as desired.

We don't give a rigorous prove of Theorem 3.1 but just a sketch.

Sketch of the proof of Theorem 3.1. If n and k are positive integers, we will denote by $\mathcal{C}_k(n)$ the set of compositions of n into k nonnegative parts. Recall that a element in $\mathcal{C}_k(n)$ is just a k-uple $(b_1, b_2, \ldots, b_k)$ of nonnegative integers such that $b_1 + b_2 + \cdots + b_k = n$. We will construct a bijection

$$f : \mathcal{N}(T, \mathbf{m}; A) \to \mathcal{C}_{m_1+1}(h_1) \times \mathcal{C}_{m_2+1}(h_2) \times \cdots \times \mathcal{C}_{m_s+1}(h_s)$$

which keeps track of the statistics ne_2 and se_2. To each $F \in \mathcal{N}(T, \mathbf{m}; A)$ we associate the sequence of compositions $(c^{(1)}, c^{(2)}, \ldots, c^{(s)})$, where for $i = 1, \ldots, s$, $c^i = (c_1^{(i)}, \ldots, c_{m_i+1}^{(i)})$ is defined by $c_1^{(i)}$ (resp., $c_j^{(i)}$ for $j = 2 \ldots m_i$, $c_{m_i+1}^{(i)}$) is the number of uncolored cells to the left of the first 1 (resp., between the j-th 1 and the $(j+1)$-th 1, to the right of the last 1) of R_i, the i-th row of the coloring of F. An example is given below.

In order to show that f is bijective, we describe its reverse g. Let $\mathbf{c} = (c^{(1)}, c^{(2)}, \ldots, c^{(s)})$ in $\mathcal{C}_{m_1+1}(h_1) \times \mathcal{C}_{m_2+1}(h_2) \times \cdots \times \mathcal{C}_{m_s+1}(h_s)$.

Then define the filling $g(\mathbf{c})$ by the following process. We start with the polyomino (empty filling) T.

(1) Color the columns indexed by the set A. We denote by F_0 the result.

(2) Suppose $R_{i_1} \prec R_{i_2} \prec \cdots \prec R_{i_s}$. For j from 1 to s, the (colored) filling F_j is obtained from F_{j-1} by the following process:

- if $m_{i_j} = 0$, then do nothing,

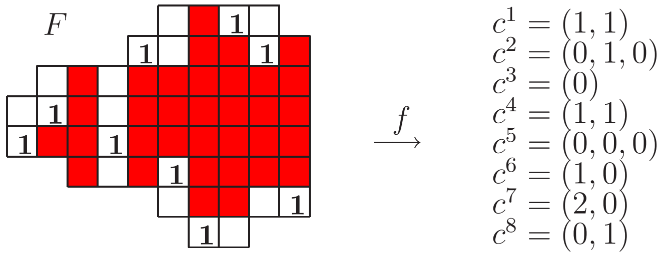


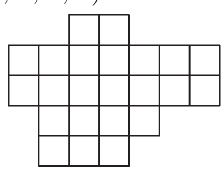
FIGURE 6. The mapping f

- else, fill the i_j-th row of F_{j-1} with m_{i_j} 1's in such a way that the number of uncolored cells strictly
  - to the left of the first 1 is $c_1^{(i_j)}$,
  - between the u-th 1 and the $(u+1)$-th 1, $1 \le u \le m_{i_j} - 1$, is $c_{u+1}^{(i_j)}$,
  - to the right of the last 1 is $c_{m_{i_j}+1}^{(i_j)}$.

  Next, color the cells which are below (resp., above) the new 1's and contained in the i_j-th rectangle if $R_{i_j} \in Up(T)$ (resp., $R_{i_j} \in Low(T)$).

(3) Set $g(\mathbf{c}) = F_s$.

For a better understanding, we give an example. Suppose T is the moon polyomino given below, $A = \{2\}$ and $\mathbf{m} = (1, 2, 1, 0, 1)$.



Suppose $\mathbf{c} = (c^{(1)}, c^{(2)}, c^{(3)}, c^{(4)}, c^{(5)})$ with $c^{(1)} = (1, 0)$, $c^{(2)} = (1, 0, 1)$, $c^{(3)} = (0, 0, 0)$, $c^{(4)} = (0)$ and $c^{(5)} = (0, 0)$. The step by step construction of $g(\mathbf{c})$ goes as follows.

It is not difficult to prove that g is the reverse of f, and thus f is bijective.

Now let $\mathbf{c} = (c^{(1)}, c^{(2)}, \ldots, c^{(s)}) \in \mathcal{C}_{m_1+1}(h_1) \times \mathcal{C}_{m_2+1}(h_2) \times \cdots \times \mathcal{C}_{m_s+1}(h_s)$ and $F = g(\mathbf{c})$ be the corresponding 01-filling. Let i be an integer in $[s]$ and ce be the cell of the i-th row R_i of F which contains the j-th 1 of R_i. It then follows from the definition of g that

$$\mathrm{luc(ce;F)} = c_1 + c_2 + \cdots + c_j \ ,$$
$$\mathrm{ruc(ce;F)} = c_{j+1} + c_{j+2} + \cdots + c_{m_i+1} = h_i - (c_1 + c_2 + \cdots + c_j).$$

Applying Proposition 2.1 and after some elementary manipulations of p, q-calculus, we get the desired result.

Remark 3.1. *Let $c = (c_1, c_2, \ldots, c_k)$ be a composition. Define the* reverse *$rev(c)$ of c as the composition $rev(c) = (c_k, c_{k-1}, \ldots, c_1)$. Given a sequence of compositions $\mathbf{c} = (c^{(1)}, c^{(2)}, \ldots, c^{(s)})$, set $\phi(\mathbf{c}) = (rev(c^{(1)}), rev(c^{(2)}), \ldots, rev(c^{(s)}))$. Then one can check that the involution Φ can be factorized as follows:*

$$\Phi = g \circ \phi \circ g^{-1}.$$

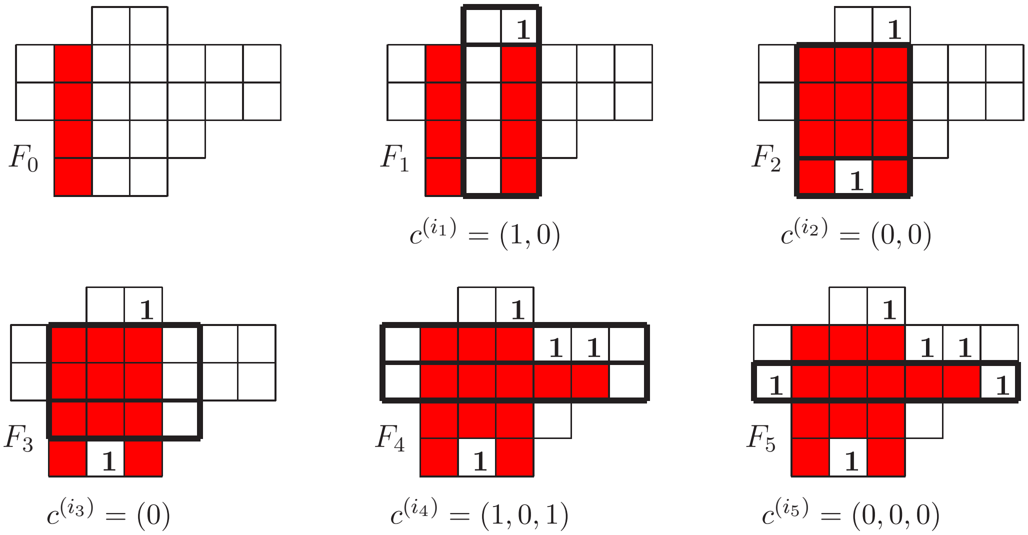


FIGURE 7. The step-by-step construction of $g(\mathbf{c})$

Remark 3.2. *It follows from Theorem 3.1 that the number of* 01*-fillings with at most one* 1 *in each column of a given moon polyomino, such that the number of ascents is k and the number of descents is ℓ, does not depend on the order of the rows, given that the resulting polyomino is again a moon polyomino.*

4. Application: symmetry of crossings and nestings over linked partitions, set partitions and matchings

Let G be a graph on $[n]$. The multiset of lefthand (resp., righthand) endpoints of the arcs of G will be denoted by left(G) (resp., right(G)). For example, if G is the graph given in Figure 1, we have $\text{left(G)} = \{1, 2, 2, 3, 5, 6, 6, 9\}$ and $\text{right(G)} = \{3, 4, 6, 7, 9, 9, 10, 11\}$. Suppose F is the 01-filling of the triangular shape Δ_n associated to G. Then it is easy to see that the number of 1's in the column (resp., row) labelled i is equal to the multiciplicity of i in left(G) (resp., right(G)). See Figure 1 for an example. As explained in the introduction, results on 2-crossings and 2-nestings in simple graphs translate into results on ascents and descents in fillings of Δ_n and vice-versa. It is then easy to derive the following result from Theorem 1.1.

If S and T are two multisubsets of $[n]$, denote by $\mathcal{G}_n(S, T)$ the set of graphs G on $[n]$ satisfying $\text{left(G)} = \text{S}$ and $\text{right(G)} = \text{T}$.

Theorem 4.1. *For any pair (S, T) of multisubsets of $[n]$ such that*

(1) *either all elements of S have multiplicity* 1*,*
(2) *either all elements of T have multiplicity* 1*,*

the joint statistic $(\text{cros}_2, \text{nest}_2)$ *is symmetrically distributed over $\mathcal{G}_n(S, T)$.*

It is now easy to recover (1.1).

4.1. **Linked partitions.** Let E and F be two finite subsets of integers. We say that E and F are *nearly disjoint* if for every $i \in E \cap F$, one of the following holds:

(a) $i = \min(E)$, $|E| > 1$ and $i \neq \min(F)$, or

(b) $i = \min(F)$, $|F| > 1$ and $i \neq \min(E)$.

A *linked partition* (see [2]) of $[n]$ is a collection of nonempty and pairwise nearly disjoint subsets whose union is $[n]$. The set of all linked partitions of $[n]$ will be denoted by $\mathcal{LP}_n$. The linear representation G_π of a linked partition $\pi \in \mathcal{LP}_n$ is the graph on $[n]$ where i and j are connected by an arc if and only if j lies in a block B with $i = \mathrm{Min(B)}$. An illustration is given in Figure . Clearly, this establishes a bijection between linked set partitions and simple graphs G such that all elements of right(G) have multiplicity one.

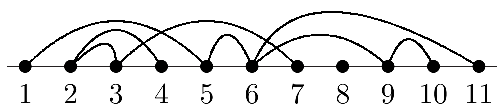


FIGURE 8. Linear representation of $\pi = \{1,5\}\{2,3,4\}\{3,7\}\{5,6\}\{6,9,11\}\{8\}\{9,10\}$

For $\mathcal{O}, \mathcal{C} \subseteq [n]$ two multisubsets of $[n]$, denote by $\mathcal{LP}_n(\mathcal{O},\mathcal{C})$ the set $\{\pi \in \mathcal{LP}_n : \mathrm{left}(\mathrm{G}_\pi) = \mathcal{O}\,,\ \mathrm{right}(\mathrm{G}_\pi) = \mathcal{C}\}$. We then derive from Theorem 4.1 the following result due to Chen et al [2].

Corollary 4.2. *For any integer n and for any pair $(\mathcal{O},\mathcal{C})$ of multisubsets of $[n]$, we have*

$$\sum_{\pi \in \mathcal{LP}_n(\mathcal{O},\mathcal{C})} p^{\mathrm{cros}_2(\mathrm{G}_\pi)} q^{\mathrm{nest}_2(\mathrm{G}_\pi)} = \sum_{\pi \in \mathcal{LP}_n(\mathcal{O},\mathcal{C})} p^{\mathrm{nest}_2(\mathrm{G}_\pi)} q^{\mathrm{cros}_2(\mathrm{G}_\pi)}.$$

Summing over all $(\mathcal{O},\mathcal{C})$, we get

$$\sum_{\pi \in \mathcal{LP}_n} p^{\mathrm{cros}_2(\mathrm{G}_\pi)} q^{\mathrm{nest}_2(\mathrm{G}_\pi)} = \sum_{\pi \in \mathcal{LP}_n} p^{\mathrm{nest}_2(\mathrm{G}_\pi)} q^{\mathrm{cros}_2(\mathrm{G}_\pi)}.$$

4.2. **Matchings and set partitions.** Recall that a (*set*) *partition* of $[n]$ is a collection of nonempty pairwise-disjoint sets, called blocks, whose union is $[n]$. A (*complete*) *matching* is just a partition whose each block contains exactly two elements. In fact, set partitions and matchings are just particular linked partitions. The set of all set partitions and matchings of $[n]$ will be denoted respectively by $\mathcal{P}_n$ and $\mathcal{M}_n$. Set partitions (and thus matchings) have a natural graphical representation, called *standard repesentation.* To each set partition π of $[n]$, one associates the graph St_π on $[n]$ whose edge set consists of arcs joining the elements of each block in numerical order. An illustration is given in Figure 8. This establishes a bijection between set partitions and simple graphs G such that all elements of right(G) and left(G) have multiplicity one. This also establishes a bijection matchings and simple graphs G such that $\mathrm{right(G)} \cap \mathrm{left(G)} = \emptyset$, $\mathrm{right(G)} \cup \mathrm{left(G)} = [\mathrm{n}]$ and all elements of right(G) and left(G) have multiplicity one. Note that for $M \in \mathcal{M}_n$, we have $St_M = G_M$ but if $\pi \in \mathcal{P}_n$ contains a block of length at least 3 then $St_\pi \neq G_\pi$.

FIGURE 9. Standard representations of $M = \{1,5\}\{2,8\}\{3,7\}\{4,10\}\{6,9\}$ and $\pi = \{1,9,10\}\{2,3,7\}\{4\}\{5,6,11\}\{8\}$

For $\mathcal{O},\mathcal{C}\subseteq[n]$ two subsets of $[n]$, let $\mathcal{P}_n(\mathcal{O},\mathcal{C})$ be the sets of set partitions $\{\pi\in\mathcal{P}_n : \mathrm{left}(\mathrm{G}_\pi)=\mathcal{O}\,,\ \mathrm{right}(\mathrm{G}_\pi)=\mathcal{C}\}$ and $\mathcal{M}_n(\mathcal{O},\mathcal{C})$ the set of matchings $\{M\in\mathcal{M}_n : \mathrm{left}(\mathrm{G_M})=\mathcal{O}\,,\ \mathrm{right}(\mathrm{G_M})=\mathcal{C}\}$. Then we derive immediately from Theorem 4.1 (or Corollary 4.2) the following results whose first of them is due to Kasraoui and Zeng [7] and second to Klazar and Noy [8].

Corollary 4.3. *For any integer n and pair $(\mathcal{O},\mathcal{C})$ of subsets of $[n]$, we have*

$$\sum_{\pi\in\mathcal{P}_n(\mathcal{O},\mathcal{C})} p^{\mathrm{cros}_2(\mathrm{St}_\pi)}q^{\mathrm{nest}_2(\mathrm{St}_\pi)}=\sum_{\pi\in\mathcal{P}_n(\mathcal{O},\mathcal{C})} p^{\mathrm{nest}_2(\mathrm{St}_\pi)}q^{\mathrm{cros}_2(\mathrm{St}_\pi)}.$$

Summing over all $(\mathcal{O},\mathcal{C})$, we get

$$\sum_{\pi\in\mathcal{P}_n} p^{\mathrm{cros}_2(\mathrm{St}_\pi)}q^{\mathrm{nest}_2(\mathrm{St}_\pi)}=\sum_{\pi\in\mathcal{P}_n} p^{\mathrm{nest}_2(\mathrm{St}_\pi)}q^{\mathrm{cros}_2(\mathrm{St}_\pi)}.$$

Corollary 4.4. *For any integer n and pair $(\mathcal{O},\mathcal{C})$ of subsets of $[n]$, we have*

$$\sum_{M\in\mathcal{M}_n(\mathcal{O},\mathcal{C})} p^{\mathrm{cros}_2(\mathrm{St_M})}q^{\mathrm{nest}_2(\mathrm{St_M})}=\sum_{M\in\mathcal{M}_n(\mathcal{O},\mathcal{C})} p^{\mathrm{nest}_2(\mathrm{St_M})}q^{\mathrm{cros}_2(\mathrm{St_M})}.$$

Summing over all $(\mathcal{O},\mathcal{C})$, we get

$$\sum_{M\in\mathcal{M}_n} p^{\mathrm{cros}_2(\mathrm{St_M})}q^{\mathrm{nest}_2(\mathrm{St_M})}=\sum_{M\in\mathcal{M}_n} p^{\mathrm{nest}_2(\mathrm{St_M})}q^{\mathrm{cros}_2(\mathrm{St_M})}.$$

4.3. **Distribution of** $(\mathrm{cros}_2,\mathrm{nest}_2)$ **over** $\mathcal{LP}_n(\mathcal{O},\mathcal{C})$**,** $\mathcal{P}_n(\mathcal{O},\mathcal{C})$ **and** $\mathcal{M}_n(\mathcal{O},\mathcal{C})$**.** We just want to point briefly that the distribution of $(\mathrm{cros}_2,\mathrm{nest}_2)$ over $\mathcal{LP}_n(\mathcal{O},\mathcal{C})$, $\mathcal{P}_n(\mathcal{O},\mathcal{C})$ and $\mathcal{M}_n(\mathcal{O},\mathcal{C})$ can be easily derived from Theorem 3.1 and the correspondence between simple graphs $[n]$ and 01-fillings of Δ_n.

Let $(\mathcal{O},\mathcal{C})$ be a pair of multisubsets of $[n]$ and denote by m_i the multiplicity of $i\in\mathcal{O}$. Also for any $i\in\mathcal{O}$ set $h_i=|\{j\in\mathcal{C}\,|\,j>i\}|-|\{j\in\mathcal{O}\,|\,j>i\}|$. Then it can be deduced from Theorem 3.1 that

$$\sum_{\pi\in\mathcal{LP}_n(\mathcal{O},\mathcal{C})} p^{\mathrm{cros}_2(\mathrm{G}_\pi)}q^{\mathrm{nest}_2(\mathrm{G}_\pi)}=\prod_{i\in\mathcal{O}}\begin{bmatrix}h_i\\ m_i\end{bmatrix}_{p,q}. \tag{4.1}$$

Note that the above identity is equivalent to a result of Chen et al [2, Theorem 3.5]. In set partitions, $m_i\le 1$ for any $i\in\mathcal{O}$. We then get the following result which is implicit in [7, Section 4]

$$\sum_{\pi\in\mathcal{P}_n(\mathcal{O},\mathcal{C})} p^{\mathrm{cros}_2(\mathrm{G}_\pi)}q^{\mathrm{nest}_2(\mathrm{G}_\pi)}=\prod_{i\in\mathcal{O}}[h_i]_{p,q}. \tag{4.2}$$

5. Concluding remarks

5.1. It is natural, in view of Theorem 1.2, to ask if the symmetry of the joint distribution of the bi-statistic $(\mathrm{ne}_2,\mathrm{se}_2)$ extend for arbitrary 01-fillings of moon polyominoes, i.e., no restrictions on the number of 1's in columns and rows. The answer is no by means of the following result.

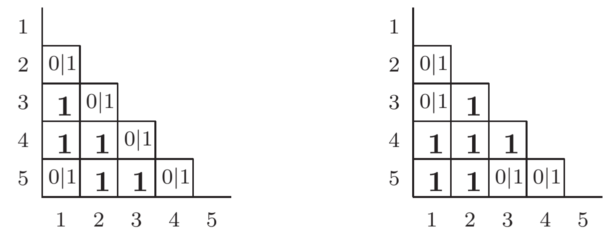
Proposition 5.1. *For any $n\ge 5$ the numbers of arbitrary 01-fillings of Δ_n*

- *with exactly $\binom{n}{4}$ descents is equal to 2^n,*

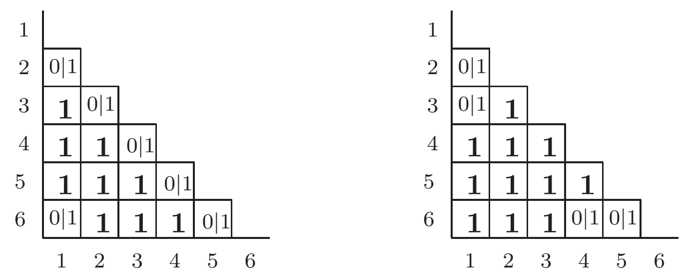
- *with exactly $\binom{n}{4}$ ascents is equal to* 16.

This implies that the joint distribution of $(\mathrm{ne}_2, \mathrm{se}_2)$ *over all arbitrary* 01*-fillings of* Δ_n *is not symmetric for any* $n \geq 5$.

Proof. We give the proof for $n = 5, 6$ since the reasoning can be generalized for arbitrary n. Suppose $n = 5$. Then one can check that the arbitrary 01-fillings of Δ_5 with exactly 5 descents and those with exactly 5 ascents have the following "form"



from which it is easy to obtain the result. Similarly, for $n = 6$, the arbitrary 01-fillings of Δ_6 with exactly 15 descents and those with exactly 15 ascents have the following "form".



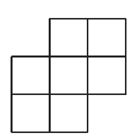
□

We then propose the following problem.

Problem 5.2. *Characterize the moon polyomino T for which the joint distribution of* $(\mathrm{ne}_2, \mathrm{se}_2)$ *over arbitrary* 01*-fillings of T is symmetric?*

Note that $T = \Delta_4$ satisfies the above condition.

5.2. One can also ask if Theorem 1.1 and Theorem 1.2 can be extended to arbitrary larger classes of polyominoes. We note that the condition of intersection free is necessary. Indeed, the polyomino T represented below is convex but not intersection free,



and

$$\sum_{F \in \mathcal{N}(T,(1,1,1),\emptyset)} p^{\mathrm{ne}_2(\mathrm{F})} q^{\mathrm{se}_2(\mathrm{F})} = \sum_{F \in \mathcal{N}(T,(1,1,1))} p^{\mathrm{ne}_2(\mathrm{F})} q^{\mathrm{se}_2(\mathrm{F})} = p^2 + 2q$$

is not symmetric. Also note that one can check that $\sum_{\mathbf{m}} \sum_{F \in \mathcal{N}(T,\mathbf{m})} p^{\mathrm{ne}_2(\mathrm{F})} q^{\mathrm{se}_2(\mathrm{F})}$ is not symmetric.

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